→ Notes II: Conservation and Symmetry, Scaling, Virial Theorems

Point:

i.) Conservation laws result from symmetry.

ii.) Symmetry → invariance L w/r Σ, λ (parameter)
→ conserved quantity

Obvious list:
- Energy → time translation invariance
- Momentum → translation invariance
- angular momentum → rotation invariance

a.) Energy

Consider time translation $t \to t + \delta t$

$$\delta L = L(t+\delta t) - L(t) \cong \delta t(\partial L/\partial t)$$

$$\delta L = 0 \Leftrightarrow \partial L/\partial t = 0$$

i.e. absence of explicit time dependence in L.

but

$$\frac{dL}{dt} = \cancel{\frac{\partial L}{\partial t}} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

and Lagranges Eqns $\Rightarrow$ $\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$

$$\frac{dL}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$

$\Rightarrow$

$$\boxed{\begin{array}{l} \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0 \quad \underline{\text{so}} \\ E = \text{energy} = \dot{q}\frac{\partial L}{\partial \dot{q}} - L \quad \text{is} \\ \text{conserved for } \partial_t L = 0. \\ \text{Defines conservative system.} \end{array}}$$

Note: Will see:

- H defined s/t

$$H = p\dot{q} - L \qquad , \quad p = \frac{\partial L}{\partial \dot{q}}$$

and eliminate $\dot{q}$ in terms p.

- in general $H \neq E$

- can define H if $\partial_t L \neq 0$ and energy not conserved.

Hamiltonian $\neq$ ~~…~~ energy conservation.

b.) Linear Momentum

→ for closed system, homogeneity of space ⇒ mechanical properties unchanged by displacement of system in space.

i.e. physics invariant upon $\underline{r} \to \underline{r} + \underline{\epsilon}$

$$\delta L = \sum_i \frac{\partial L}{\partial \underline{r}_i} \cdot \underline{\epsilon} = \underline{\epsilon} \cdot \sum_i \frac{\partial L}{\partial \underline{r}_i}$$

so LEOM $\Rightarrow$ $$\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = 0$$

$\partial L / \partial \underline{v}_i \equiv p_i$ $\underline{\underline{\text{so}}}$

$$\frac{d}{dt}\left(\sum_i \underline{p}_i\right) = \frac{d}{dt}\underline{P} = 0$$

- momentum conservation follows from spatial homogeneity

- applies by component.

- $\underline{p}_i$ = cons if $\underline{F}_i = -\frac{\partial U}{\partial \underline{r}_i} = \frac{\partial L}{\partial \underline{r}_i} = 0$

$\downarrow$
generalized force.

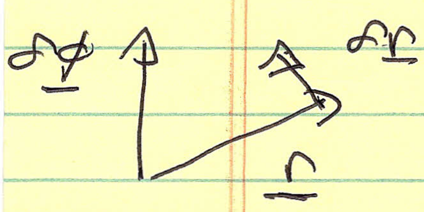
c.) <u>Angular Momentum</u>

$\rightarrow$ isotropy of space for closed system $\Rightarrow$ physics invariant under infinitesimal rotation

$\delta \underline{\phi} \equiv$ (vector) infinitesimal rotation

$\dfrac{\delta \underline{\phi}}{|\delta \underline{\phi}|} \Rightarrow$ axis direction

$|\delta \underline{\phi}| \rightarrow$ magnitude.



$$\delta \underline{r} = \delta \underline{\phi} \times \underline{r}$$

$$\delta \underline{v} = \delta \underline{\phi} \times \underline{v}$$

so isotropy upon rotation $\Rightarrow L$ invariant, $\delta L = 0$,

$\Rightarrow$

$$\delta L\Big|_{rot.} = \sum_i \left(\frac{\partial L}{\partial \underline{r}_i} \cdot \delta \underline{r}_i + \frac{\partial L}{\partial \underline{v}_i} \cdot \delta \underline{v}_i \right) = 0$$

$$= \sum_i \left(\dot{\underline{p}}_i \cdot \delta \underline{\phi} \times \underline{r}_i + \underline{p}_i \cdot \delta \underline{\phi} \times \underline{v}_i \right)$$

re-arranging

$$\delta L\Big|_{rot} = 0 = \delta \underline{\phi} \cdot \left(\sum_i \underline{r}_i \times \dot{\underline{p}}_i + \dot{\underline{r}}_i \times \underline{p}_i \right)$$

$$= \delta \underline{\phi} \cdot \frac{d}{dt} \left[\sum_i \left(\underline{r}_i \times \underline{p}_i \right) \right]$$

$$\delta L\Big|_{rot} = 0 \Rightarrow \frac{d}{dt} \underline{L} = 0 .$$

$$\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$$

$\therefore$ rotational invariance $\Rightarrow$ angular momentum conserved.

Note: Angular momentum depends on choice of origin, except when system is at rest, as a whole.

$$\underline{L} = \sum_i \underline{r}_i \times \underline{p}_i$$

$$\underline{r} \to \underline{r}' + \underline{a}$$

$$\underline{L} = \sum_i \underline{r}_i' \times \underline{p}_i + \sum_i \underline{a} \times \underline{p}_i$$

$$= \underline{L}' + \underbrace{a \times \underline{P}}_{\text{origin dep, unless } \underline{P} = 0}.$$

Thus, summing it all up:

- time homogeneity $\to$ $\delta L = 0$ upon $t \to t + \delta t$

$\Rightarrow \partial_t L = 0 \Leftrightarrow$ <u>energy conservation</u>

- spatial homogeneity $\Rightarrow$ $\delta L = 0$
  $\underline{x} \to \underline{x} + \underline{\delta x}$

$\partial_x L = 0 \Rightarrow$ linear momentum conservation

- rotational isotropy $\Rightarrow$ $\delta L = 0$
  $\underline{\phi} \to \underline{\phi} + \underline{\delta \phi}$

$\partial L / \partial \phi = 0 \Rightarrow$ angular momentum conserved.

$\Rightarrow$ <u>Connection</u>:

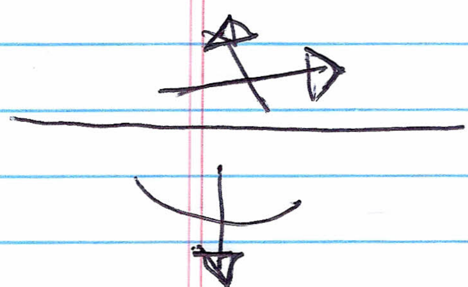
symmetry $\rightarrow$ ignorable or 'cyclic' coordinate $\rightarrow$ conservation law

$\boxed{\text{Noether's Theorem}}$

→ Symmetry Exercises (LxL / Problem 3 pg. 21)

? which components of $\underline{P}$, $\underline{L}$ conserved in following fields.
(think as U=const zones)

a.) infinite homogeneous plate



- 2 components $\underline{P}$ in plane // plate (P_x, P_y)
- $L_\perp$ component $\perp$ to plate plane (L_z)

b.)

infinite homogeneous cylinder

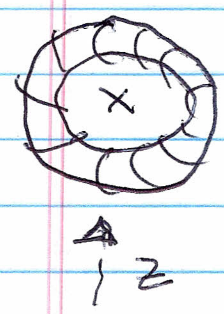
- $\underline{P}$ component along axis of cylinder (P_z)
- $\underline{L}$ component along axis of cylinder (L_z)

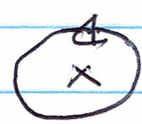
c.) infinite homogeneous rectangular prism

— p_z parallel to prism.

d.) Torus

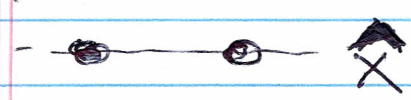
L_z only





— L_z only (i.e. toroidal angular momentum)

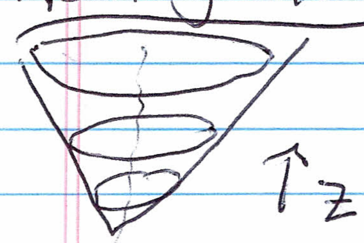
e.) Two points



— L_x

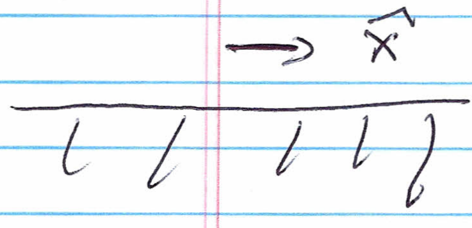
(angular momentum for rotations about axis of line joining 2 pts.).

f.) homogeneous cone



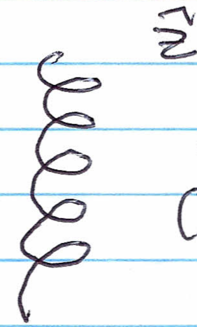
— L_z, only

g.) infinite half plate (2D)



$-\ p_x$ only

h.) infinite homogeneous cylindrical helix, pitch h



$h \equiv$ pitch (vertical distance in 1 rotation)

Invariance:

$\delta\phi$ rotation $+ \frac{\delta\phi}{2\pi} h$ translation

$$\delta L = \frac{\partial L}{\partial z}\,\delta z + \frac{\partial L}{\partial \phi}\,\delta\phi$$

$$dz = \frac{h}{2\pi}\,\delta\phi$$

$$\partial L/\partial z = d/dt\,(p_z), \quad \partial L/\partial \phi = \frac{d}{dt}(L_z)$$

so

$$\delta L = \dot{p}_z\left(\frac{h}{2\pi}\delta\phi\right) + \dot{L}_z\,\delta\phi$$

$$= \delta\phi\left(h/2\pi\,\dot{p}_z + \dot{L}_z\right)$$

so $\delta L = 0 \Rightarrow \frac{d}{dt}\left(\frac{h}{2\pi} p_z + L_z\right) = 0$

$$\frac{h}{2\pi} p_z + L_z = \text{const.}$$

now, ...

What, exactly, is the Theorem?

Precise statement of Noether's Thm:

If the functional:

$$S = \int_a^b L(t, q^\mu, \dot{q}^\mu)\,dt$$

is an extremal,

and if under the infinitesimal transformation:

$$t' = t + \epsilon\tau + \ldots$$

$$q'^\mu = q^\mu + \epsilon\rho^\mu + \ldots$$

the functional is invariant according to:

$$L' \frac{dt'}{dt} - L = \epsilon \frac{dF}{dt} + h.o. \ldots$$

<u>then</u>:

$$p_\mu Q^\mu - HT - F = \text{const}$$

is a conservation law.

Re: scale symmetry $\Rightarrow$

if $U(\alpha \underline{r}_1, \alpha \underline{r}_2, \ldots \alpha \underline{r}_n) = \alpha^k U(\underline{r}_1, \underline{r}_2, \ldots \underline{r}_n)$

i.e. re-scale variables

$\Rightarrow$ U is $\boxed{\text{homogeneous.}}$ (related to scale invariance)

Many relevant U are homogeneous, i.e.

harmonic oscillator: $k = 2$ rescale variables $\rightarrow$ same U

Coulomb / gravity: $k = -1$

etc. $\Rightarrow$ homogeneous $U \leftrightarrow$ power law structure.

Now, recall:

EOM from $\delta \int dt\, L = 0$, so if

$L \rightarrow \sigma L$, EOM unchanged. !

σ const. factor

$\Rightarrow$ [Multiplying Lagrangian by constant factor leaves physics unchanged.

Next:

→ Scale Symmetry and Virial Theorems
i.e. (rescalings)

- scale symmetry of interest,
as well space-time symmetry.

- scale symmetry ↔ how exploit
dimensional analysis in framework
of Lagrangian mechanics.

- scale symmetry ↔ related to
virial theorems → useful in
Astrophysics.

Scale Symmetry / Virial Theorems

→ where dimensional analysis
meets symmetry constraints.

→ So, for homogeneous U, can generate class of rescalings which multiply Lagrangian by const. factor

⇒ same EOM!

→ $\boxed{\text{such rescalings define basic class of relations between quantities.}}$

→ useful for basic story/characteristics w/o detailed crank. → insight!

Now: $$S = \int dt \left(\tfrac{1}{2} m \dot{\underline{r}}^2 - U(\underline{r}) \right)$$

$$r \to \alpha r'$$

$$t \to \beta t'$$

$$S = \int dt \left(\tfrac{1}{2} m \frac{\alpha^2}{\beta^2} \dot{r}'^2 - \alpha^k U(r') \right)$$

so if $\boxed{\alpha^2/\beta^2 \sim \alpha^k}$ ⇒ L multiplied by net factor and δS invariant!

$\alpha \to$ space re-scaling

$$\boxed{\beta \sim \alpha^{1-k/2} \text{ defines space-time rescaling leaving EOM unchanged}}$$

$$\alpha \sim l'/l \Rightarrow \beta \sim (t'/t) \sim (l'/l)^{1-k/2}.$$

homogeneous only.

Equivalently:

a) V'/V

$$S = \int dt \left(\frac{m}{2}\dot{r}^2 - U\right) = \int dt \left(\frac{mv^2}{2} - U\right)$$

U homogeneous: $\left\{ U(\alpha r) \to \alpha^k U \text{ upon } r \to \alpha r \right.$

$$\Rightarrow V'/V \sim \alpha^{k/2} \sim (l'/l)^{k/2}$$

b) $E'/E \sim (l'/l)^k$

c) $L'/L \sim (l'/l)^{1+k/2}$.

Examples

→ $U \sim z$, as in uniform gravity

$k \sim 1$

Fall time scaling?

$$t'/t \sim (\ell'/\ell)^{1-k/2} \sim (\ell'/\ell)^{1/2}$$

$$T \sim \sqrt{z}$$

→ $U \sim r^2$ $\underline{k=2}$ h.o.

Period vs distance/amplitude?

$$t'/t \sim \alpha^{1-k/2} \sim (\ell'/\ell)^0$$

Period indep. amplitude

→ $U \sim r^{-1}$ $k=-1$ (Coulomb)

$$t'/t \sim (\ell'/\ell)^{1-k/2} \sim (\ell'/\ell)^{3/2}$$

$T \sim r^{3/2}$ ⇒ Kepler's 3rd Law

More Examples

→ Ratio of times for particles on same path, with PE differing by a factor (const mult.), same mass

$$S = \int dt \left(\frac{m \dot{r}^2}{2} - U \right)$$

$$= \int dt \left(\frac{m}{2} \frac{\alpha^2}{\beta^2} \dot{r}'^2 - \frac{U'}{U} U \right)$$

$\alpha = 1 \Rightarrow$ same path ✓

$\underline{\text{so}}$ $\beta^{-2} = U'/U \Rightarrow t'/t \sim \sqrt{U/U'}$

→ same U, m m'

$$\frac{m'}{\beta^2} = m \qquad \frac{t'}{t} \sim \sqrt{m'/m}$$

→ Homogeneous Potential brings us to Virial Theorem !

What is the virial theorem ? ⇒ [Reln. re avg.

– consider a system of particles

– examine :

$$\frac{d}{dt}\Big(\underbrace{\sum_i \underline{p}_i \cdot \underline{x}_i}_{\text{action}}\Big) = \sum_i \underline{p}_i \cdot \dot{\underline{x}}_i + \sum_i \dot{\underline{p}}_i \cdot \underline{x}_i$$

$$= 2T - \sum_i \frac{\partial U}{\partial \underline{x}_i} \cdot \underline{x}_i$$

Now, time average

$$\left\langle \frac{d}{dt}\Big(\sum_i \underline{p}_i \cdot \underline{x}_i\Big) \right\rangle$$ → time avg. of deriv.

where $\langle A \rangle = \frac{1}{\tau}\int_0^{\tau} A \quad , \tau \to \infty .$

Now if $\sum_i \underline{p}_i \cdot \underline{x}_i$ bounded in time

$$\left\langle \frac{d}{dt} \sum_i \underline{p}_i \cdot \underline{x}_i \right\rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \left(\frac{d}{dt} \left(\sum_i \underline{p}_i \cdot \underline{x}_i \right) \right)$$

$$= \frac{1}{T} \sum_i \underline{p} \cdot \underline{x}_i \Big|_0^T$$

$\to 0$, if bounded.

then

$$0 = 2\langle T \rangle - \sum_i \frac{\partial U_i}{\partial \underline{x}_i} \cdot \underline{x}_i$$

so

$$2\langle T \rangle = \left\langle \sum_i \frac{\partial U_i}{\partial \underline{x}_i} \cdot \underline{x}_i \right\rangle$$

Now, if $U = U(\underline{x}_1, \underline{x}_2, \ldots \underline{x}_n)$ homogeneous

$$U(\alpha \underline{x}_1, \ldots \alpha \underline{x}_n) = \alpha^k U$$

$\Rightarrow$

$$\boxed{2\langle T \rangle = k \langle U \rangle}$$

deriv $\to$ power $\to$ k

and also

$$T + U = E = \langle T \rangle + \langle U \rangle$$

so

$$2\langle T \rangle = k \langle U \rangle$$

$$E = \langle T \rangle + \langle U \rangle$$

$\Rightarrow$

$$\langle U \rangle = \frac{2}{k+2} E$$

$$E = \frac{(k+2)}{k} \langle T \rangle$$

check:

(n.b. must have bound state for time avg. to converge).

- $k = -1$ (gravity)

$$E = -\langle T \rangle$$

$$\langle U \rangle = E$$

Total energy negative for bound (gravitationally) cluster

- $k = 2$ h.o.

$\langle U \rangle = E/2$, $\langle T \rangle = E/2$ ✓.

Why Care?

[measure $\langle T \rangle$, use U
get E

- can measure $\langle T \rangle$ by spectroscopy, etc. then relate to Energies

- of course, virial thm relates energies to potential structure.

and

- computing virial / virial avg. is a single # characterizing a cluster of particles

$$f(\underline{x}, \underline{v}, t) \rightarrow V(\underline{x}, t) \rightarrow \langle T \rangle, \langle U \rangle, E$$

dist. — fluid field — #

Boltzmann $\rightarrow$ fluid $\rightarrow$ virial

$\wr$ Velocity Moment — $\downarrow$ $\sum_i$, $\int dx$.